

On a periodic solution of the focusing nonlinear Schrödinger equation

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Abstract

A periodic two-phase algebro-geometric solution of the focusing nonlinear Schrödinger equation is constructed in terms of elliptic Jacobi theta-functions. A dependence of this solution on the parameters of a spectral curve is investigated. An existence of a real smooth finite-gap solution of NLS equation with complex initial phase is proven. Degenerations of the constructed solution to one-phase traveling wave solution and solutions in the form of the plane waves are carried.

Keywords: freak waves, nonlinear Schrödinger equation, theta function, reduction, covering

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Introduction

Currently, one of the most hot topic of nonlinear physics and mathematics is an investigation of the nature and prediction of appearance so called “rogue waves” or “freak waves”. Freak waves are very popular in last time solutions that represent local short-term grows of amplitude or “waves that appear from nowhere and disappear without a trace” [1]. Although freak waves were detected in models concerning various fields of physics (see for example [2]) main fields of they appearance are hydrodynamics [3–6] and nonlinear optics [7–9].

In general, a study of freak waves in principle approximation is based on a consideration of the focusing nonlinear Schrödinger equation (NLS)

$$ip_t + p_{xx} + 2|p|^2 p = 0, \quad i^2 = -1. \quad (1)$$

Last time physicians and mathematicians studied very actively rational solutions of (1); usually these solutions are obtained by Darboux transformation and its generalizations [10–19]. Exact solutions distinguished from rational ones are mentioned quite seldom [7, 9, 20–28].

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While taking into account more fine effects [4, 7, 28–33] equations under consideration differ from (1) by usage of the additional terms leading to non-integrability of an equation. Therefore, a construction of some classes of solutions becomes impossible. At the same time there exist such situations when a mentioned additional terms may be omitted [7, 28, 33]. In this case a consideration integrable equation (1) allows to use solution classes intrinsic to integrable equations such as multi-phase periodic solutions.

On the one hand, several works [5, 6, 8] contain statement that in the laboratory experiments some waves were observed with the characteristics close to those of rational solutions. On the other hand, the outcomes of the studies [28, 33] show that observed in experiments waves are close to multi-phase solutions. However, there is a possibility that if authors of studies [5, 6, 8] would compare experimental data with multi-phased solutions' characteristics, then they would obtain relatively close agreement. Especially it may be so in the cases when the sizes of cell of lattice of periods are considerably larger than dimensions of tank or waveguide.

Of course, one can not expect the strictly periodic rogue waves to appear on a water surface. However, in case of nonlinear optics, is quite possible to create situations leading to an appearance of periodic freak waves. For example, in the research [9] authors discuss an observation Kuznetsov-Ma soliton [34, 35] in the experiment. Let us remark that this solution is a limit case of two-phase algebro-geometric solution [36]. Moreover, usage appropriate limit in multi-phase solutions [36–39] allows us to obtain a rational, solitonic or periodic in x homoclinic solutions [28, 33], i.e., solutions that are usually obtained with the help of Darboux transformations. It would be interesting to know what multi-phase solutions lead to symmetrical rational solutions from [12, 13].

In the present paper we consider two-phase periodic in x and in t algebro-geometric solution of equation (1). Considered two-phase solution is a particular case of multi-phase solutions. Multi-phase algebro-geometric solutions are constructed by “finite-gap (algebro-geometric) integration method” [37, 40, 41]; this method was created in the works of Dubrovin, Novikov, Marchenko, Lax, McKean, van Moerbeke, Matveev, Its, Krichever [42–50]. One who is interested in details of the development of the method is to refer to the review article [51]. It should be mentioned that another method of constructing multi-phase algebro-geometric solutions of integrable nonlinear equations exists [38, 39, 52–55]. Let us remark that first method is based on Baker-Akhiezer function but second method is based on some Fuchs identities [56]. In our paper we use first method and Its' and Kotlyarov's classic formulas [23, 57] (see also [37]).

Our aim here is to understand an influence of parameters of a spectral curve on the behavior and the shape of two-phase algebro-geometric solutions of (1). The first section of this paper contains the basic notations and classic formulas of algebro-geometric solutions of the focusing nonlinear Schrödinger equation (1). The second section of the paper is devoted to the construction of an example of two-phase algebro-geometric solution; As in [58, 59], new solution is expressed in Jacobi elliptic theta functions. It should be noted that present solution differ from previous theta-functional solutions [24, 25, 58, 59]. In particular, in present

paper all branch points have one same real parts, while in [58, 59] all pairs of branch points have different real parts. As we can see, this leads to the following results. Crests of amplitude of constructed periodic solution disposed in the nodes of a quadrangular lattice. Crests of previous solutions formed two quadrangular lattice; nodes of one lattice disposed in centers of parallelograms of second lattice. Therefore, considered in present paper solution is a special case of two-phase solutions. The third section contains an analysis of dependence of solution on the parameters of a spectral curve. In particular, we show that if a distance between branching points decreases, then a steepness of front of solution increases. Also in present work we consider for the first time a dependence of two-phase solution of equation (1) from a sum of branch points. We show that if a sum of branch points differs from zero then then solution of equation (1) has not rectangular lattice of periods, and if sum equals zero then lattice is rectangular. Another new statement is following: a real smooth finite-gap solution of NLS equation may have a complex initial phase. In [37] one can found a statement about necessity of a real initial phase for smooth real finite-gap solution. In the fourth section we calculate some limits of the investigated two-phase solution. These limits include a new theta-functional expression for one-phase solution. Some technical details of related calculations can be found in Appendix.

1 Finite-gap solutions of the focusing nonlinear Schrödinger equation

We use well-known method of construction finite-gap solutions of NLS equation [23, 37, 40, 60, 61]. It is based on the fact that the equation (1) can be obtained from the coupled nonlinear Schrödinger equation (cNLS)

$$\begin{cases} ip_t + p_{xx} - 2p^2q = 0, \\ iq_t - q_{xx} + 2pq^2 = 0 \end{cases} \quad (2)$$

by reduction $q = -p^*$. In turn, the cNLS (2) results from the compatibility condition of the following system of linear equations (Lax pair)

$$\Psi_x = \mathfrak{U}\Psi, \quad (3)$$

$$\Psi_t = \mathfrak{V}\Psi, \quad (4)$$

where

$$\mathfrak{U} = \lambda \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} 0 & ip \\ -iq & 0 \end{pmatrix}, \quad \mathfrak{V} = 2\lambda\mathfrak{U} + \begin{pmatrix} -ipq & -p_x \\ -q_x & ipq \end{pmatrix}.$$

Finite-gap solutions of system (2) are parameterized by the hyper-elliptic curve $\Gamma = \{(w, \lambda)\}$ of the genus g :

$$\Gamma : \quad w^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j).$$

The branch points ($\lambda = \lambda_j$, $j = 1, \dots, 2g + 2$) of this curve are the endpoints of the spectral arcs of continuous spectrum of Dirac operator (3). Infinitely far point of the spectrum corresponds to two different points \mathcal{P}_∞^\pm on the curve Γ . If $q = -p^*$, then the curve Γ has the form

$$\Gamma : \quad w^2 = \prod_{j=1}^{g+1} (\lambda - \lambda_j)(\lambda - \lambda_j^*), \quad \Im \lambda_j \neq 0. \quad (5)$$

Following a standard procedure of constructing a finite-gap solutions [37, 50], let us to choose on Γ a canonical basis of cycles $\gamma^t = (a_1, \dots, a_g, b_1, \dots, b_g)$ with matrix of indices of intersection

$$C_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The condition $q = -p^*$ implies [37, 50] that the basis of cycles should satisfy the relation:

$$\hat{\tau}_1 \mathbf{a} = -\mathbf{a}, \quad \hat{\tau}_1 \mathbf{b} = \mathbf{b} + K \mathbf{a}, \quad (6)$$

where τ_1 is an anti-holomorphic involution

$$\tau_1 : (w, \lambda) \rightarrow (w^*, \lambda^*).$$

A normalized basis of holomorphic differentials $d\mathcal{U}_j$ corresponds to the canonical basis of cycles

$$\oint_{a_k} d\mathcal{U}_j = \delta_{kj}, \quad k, j = 1, \dots, g. \quad (7)$$

It is well known [50, 62–64] that a matrix of periods B of a curve Γ ,

$$B_{kj} = \oint_{b_k} d\mathcal{U}_j, \quad k, j = 1, \dots, g, \quad (8)$$

is a symmetric matrix with positively defined imaginary part.

Let us define g -dimensional Riemann theta function with characteristics $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathbb{R}^g$ [50, 56, 62]:

$$\begin{aligned} \Theta[\boldsymbol{\eta}^t; \boldsymbol{\zeta}^t](\mathbf{u}|B) &= \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp\{\pi i(\mathbf{m} + \boldsymbol{\eta})^t B(\mathbf{m} + \boldsymbol{\eta}) + 2\pi i(\mathbf{m} + \boldsymbol{\eta})^t(\mathbf{u} + \boldsymbol{\zeta})\}, \\ \Theta[\mathbf{0}^t; \mathbf{0}^t](\mathbf{u}|B) &\equiv \Theta(\mathbf{u}|B), \end{aligned} \quad (9)$$

where $\mathbf{u} \in \mathbb{C}^g$, and summation pass over g -dimensional integer lattice.

Let us also define on Γ normalized Abelian integrals of a second kind $\Omega_1(\mathcal{P})$, $\Omega_2(\mathcal{P})$ and third kind $\omega_0(\mathcal{P})$ with the following asymptotic at infinitely far points \mathcal{P}_∞^\pm :

$$\oint_{a_k} d\Omega_1 = \oint_{a_k} d\Omega_2 = \oint_{a_k} d\omega_0 = 0, \quad k = 1, \dots, g,$$

$$\begin{aligned}
\Omega_1(\mathcal{P}) &= \mp i (\lambda + K_1 + O(\lambda^{-1})), & \mathcal{P} &\rightarrow \mathcal{P}_\infty^\pm, \\
\Omega_2(\mathcal{P}) &= \mp i (2\lambda^2 + K_2 + O(\lambda^{-1})), & \mathcal{P} &\rightarrow \mathcal{P}_\infty^\pm, \\
\omega_0(\mathcal{P}) &= \mp (\ln \lambda - \ln K_0 + O(\lambda^{-1})), & \mathcal{P} &\rightarrow \mathcal{P}_\infty^\pm, \\
w &= \pm (\lambda^{g+1} + O(\lambda^g)), & \mathcal{P} &\rightarrow \mathcal{P}_\infty^\pm.
\end{aligned}$$

Let us denote by $2\pi i \mathbf{U}$, $2\pi i \mathbf{V}$ the vectors of b -periods of Abelian integrals of the second kind $\Omega_1(\mathcal{P})$, $\Omega_2(\mathcal{P})$ respectively.

Theorem 1 ([23]). *Solution of cNLS (2) has a form*

$$\begin{aligned}
p(x, t) &= \frac{2K_0}{A} \frac{\Theta(\mathbf{Z})\Theta(\mathbf{U}x + \mathbf{V}t + \mathbf{Z} - \mathbf{\Delta})}{\Theta(\mathbf{Z} - \mathbf{\Delta})\Theta(\mathbf{U}x + \mathbf{V}t + \mathbf{Z})} \exp\{2i\Phi(x, t)\}, \\
q(x, t) &= 2AK_0 \frac{\Theta(\mathbf{Z} - \mathbf{\Delta})\Theta(\mathbf{U}x + \mathbf{V}t + \mathbf{Z} + \mathbf{\Delta})}{\Theta(\mathbf{Z})\Theta(\mathbf{U}x + \mathbf{V}t + \mathbf{Z})} \exp\{-2i\Phi(x, t)\},
\end{aligned} \tag{10}$$

where $\Phi(x, t) = K_1x + K_2t$. Vector $\mathbf{\Delta}$ is a vector of Abelian holomorphic integrals that are calculated along the path between points \mathcal{P}_∞^- and \mathcal{P}_∞^+ , and this path does not intersect any of the basic cycles. The \mathbf{Z} is initial phase of solution, $A \neq 0$ is an arbitrary constant.

Eqs. (10), (5) imply that for the focusing NLS equation an amplitude of solution $|p|$ satisfies the equality

$$|p|^2 = -4K_0^2 \frac{\Theta(\mathbf{U}x + \mathbf{V}t + \mathbf{Z} - \mathbf{\Delta})\Theta(\mathbf{U}x + \mathbf{V}t + \mathbf{Z} + \mathbf{\Delta})}{\Theta^2(\mathbf{U}x + \mathbf{V}t + \mathbf{Z})}, \tag{11}$$

where

$$\Im \mathbf{U} = \Im \mathbf{V} = \Im \mathbf{Z} = \mathbf{0}, \quad K_0^2 < 0.$$

2 The curve of genus $g = 2$ with involution

To construct an example of solution we will use the curve Γ_2 (fig. 1); this curve is defined by equation

$$w^2 = (\lambda^2 - 2\lambda_0\lambda + |\lambda_1|^2)(\lambda^2 - 2\lambda_0\lambda + |\lambda_2|^2)(\lambda^2 - 2\lambda_0\lambda + |\lambda_3|^2), \tag{12}$$

where

$$\Re \lambda_1 = \Re \lambda_2 = \Re \lambda_3 = \lambda_0, \quad 0 < \Im \lambda_1 < \Im \lambda_2 < \Im \lambda_3.$$

Let us select the basis of cycles Γ_2 as on the fig. 1 and the basis of normalized holomorphic differentials:

$$d\mathcal{U}_j = \frac{(c_{j1}\lambda + c_{j2})d\lambda}{w}.$$

Since two holomorphic involutions exist on Γ_2 :

$$\tau_0 : (w, \lambda) \rightarrow (-w, \lambda), \tag{13}$$

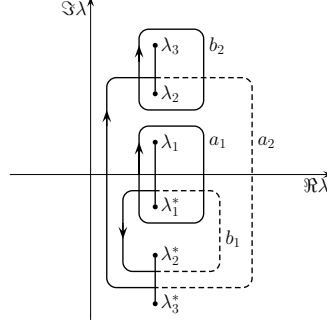


Figure 1: The curve Γ_2

$$\tau_2 : (w, \lambda) \rightarrow (w, 2\lambda_0 - \lambda), \quad (14)$$

then the curve Γ_2 covers two elliptic curves: $\Gamma_+ = \Gamma/\tau_2$ (fig. 2)

$$\Gamma_+ : \quad \chi_+^2 = (t + a^2)(t + b^2)(t + c^2) \quad (15)$$

and $\Gamma_- = \Gamma/(\tau_0\tau_2)$ (fig. 3)

$$\Gamma_- : \quad \chi_-^2 = t(t + a^2)(t + b^2)(t + c^2), \quad (16)$$

where

$$a = \Im \lambda_1, \quad b = \Im \lambda_2, \quad c = \Im \lambda_3.$$

The covering mappings are given by the formulae

$$t = (\lambda - \lambda_0)^2, \quad \chi_+ = w, \quad \chi_- = (\lambda - \lambda_0)w, \quad (17)$$

$$\frac{dt}{\chi_+} = \frac{2(\lambda - \lambda_0)d\lambda}{w}, \quad \frac{dt}{\chi_-} = \frac{2d\lambda}{w}. \quad (18)$$

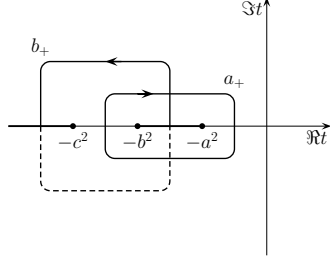


Figure 2: The curve Γ_+

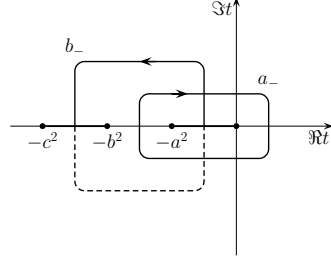


Figure 3: The curve Γ_-

The existence of covering lead [65,66] to following proposition.

Theorem 2. *Generated by the spectral curve Γ_2 two-phase solution can be expressed by elliptic theta functions. Parameters of related solution can be expressed in terms of elliptic integrals on the curves Γ_{\pm} .*

It follows from equalities (17) that covering mappings generate the following transformations between bases of cycles

$$\begin{aligned} \hat{\sigma}_+ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} a_+ \\ b_+ \end{pmatrix}, & \hat{\sigma}_- \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_- \\ b_- \end{pmatrix}, \\ \hat{\sigma}_+ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_+ \\ b_+ \end{pmatrix}, & \hat{\sigma}_- \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_- \\ b_- \end{pmatrix}. \end{aligned} \quad (19)$$

It follows from relations (18), (19) that the matrix of coefficients of normal-

ized holomorphic differentials $d\mathcal{U}_j$ equals to

$$C = \begin{pmatrix} 0 & i/(2A_-) \\ -i/(2A_+) & i\lambda_0/(2A_+) \end{pmatrix},$$

where

$$A_+ = \int_{a^2}^{b^2} \frac{dt}{\sqrt{(t-a^2)(b^2-t)(c^2-t)}}, \quad A_- = \int_0^{a^2} \frac{dt}{\sqrt{t(a^2-t)(b^2-t)(c^2-t)}}.$$

Calculating the matrix of the periods of the curve Γ_2 we have

$$B = \begin{pmatrix} i\mathfrak{b}_-/2 & -1/2 \\ -1/2 & i\mathfrak{b}_+/2 \end{pmatrix},$$

where $\mathfrak{b}_\pm = B_\pm/A_\pm$,

$$B_+ = \int_{b^2}^{c^2} \frac{dt}{\sqrt{(t-a^2)(t-b^2)(c^2-t)}}, \quad B_- = \int_{a^2}^{b^2} \frac{dt}{\sqrt{t(t-a^2)(b^2-t)(c^2-t)}}.$$

It follows from Riemann bilinear relations [50,62] that b -periods of the normalized Abelian differentials of the second kind are equal

$$U_m = -i \left(\left. \frac{d\mathcal{U}_m}{d\xi_-} \right|_{\xi_-=0} - \left. \frac{d\mathcal{U}_m}{d\xi_+} \right|_{\xi_+=0} \right) = -2ic_{m1},$$

$$V_m = -2i \left(\left. \frac{d^2\mathcal{U}_m}{d\xi_-^2} \right|_{\xi_-=0} - \left. \frac{d^2\mathcal{U}_m}{d\xi_+^2} \right|_{\xi_+=0} \right) = -12i\lambda_0 c_{m1} - 4ic_{m2},$$

where $\xi_\pm = \lambda^{-1}$ are local parameters in the neighborhood of the infinitely far points \mathcal{P}_∞^\pm , and c_{mj} are coefficients of holomorphic differentials $d\mathcal{U}_m$.

Calculating derivatives we get

$$U_1 = 0, \quad U_2 = -1/A_+, \quad V_1 = 2/A_-, \quad V_2 = -4\lambda_0/A_+.$$

Changing in formula (9) a summation with respect to \mathbf{m} on a summation with respect to \mathbf{n} and \mathbf{k} :

$$m_j = 2n_j + k_j, \quad n_j \in \mathbf{Z}, \quad k_j \in \{0; 1\}, \quad j = 1, 2$$

we obtain

$$\Theta(\mathbf{u}|B) = \vartheta_3(2u_1|2i\mathfrak{b}_-)\vartheta_3(2u_2|2i\mathfrak{b}_+) + \vartheta_2(2u_1|2i\mathfrak{b}_-)\vartheta_3(2u_2|2i\mathfrak{b}_+) + \\ + \vartheta_3(2u_1|2i\mathfrak{b}_-)\vartheta_2(2u_2|2i\mathfrak{b}_+) - \vartheta_2(2u_1|2i\mathfrak{b}_-)\vartheta_2(2u_2|2i\mathfrak{b}_+),$$

where ϑ_j are Jacoby elliptic theta functions [67]

$$\vartheta_1(u|b) = 2 \sum_{m=1}^{\infty} (-1)^{m-1} h^{(m-1/2)^2} \sin[(2m-1)\pi u], \quad h = e^{\pi ib},$$

$$\begin{aligned}\vartheta_2(u|b) &= 2 \sum_{m=1}^{\infty} h^{(m-1/2)^2} \cos[(2m-1)\pi u], \\ \vartheta_3(u|b) &= 1 + 2 \sum_{m=1}^{\infty} h^{m^2} \cos(2m\pi u), \\ \vartheta_4(u|b) &= 1 + 2 \sum_{m=1}^{\infty} (-1)^m h^{m^2} \cos(2m\pi u).\end{aligned}$$

Therefore, associated with curve (12) solution has the form

$$p = -2iK_0 \frac{H(u_1 + i\delta, u_2 + 1)}{H(u_1, u_2)} \exp\{2iK_1x + 2iK_2t\}, \quad (20)$$

$$|p|^2 = -4K_0^2 \frac{H(u_1 - i\delta, u_2 - 1)H(u_1 + i\delta, u_2 + 1)}{H^2(u_1, u_2)}, \quad (21)$$

where $u_1 = \kappa_1 t + 2Z_1$, $u_2 = kx + \kappa_2 t + 2Z_2$,

$$\begin{aligned}\kappa_1 &= 4/A_-, \quad k = 2/A_+, \quad \kappa_2 = 8\lambda_0/A_+, \quad \delta = B_-^1/A_-, \\ B_-^1 &= \int_{c^2}^{\infty} \frac{dt}{\sqrt{t(t-a^2)(t-b^2)(t-c^2)}}, \\ H(u_1, u_2) &= \vartheta_3(u_1|2i\mathfrak{b}_-)\vartheta_3(u_2|2i\mathfrak{b}_+) + \vartheta_2(u_1|2i\mathfrak{b}_-)\vartheta_3(u_2|2i\mathfrak{b}_+) + \\ &\quad + \vartheta_3(u_1|2i\mathfrak{b}_-)\vartheta_2(u_2|2i\mathfrak{b}_+) - \vartheta_2(u_1|2i\mathfrak{b}_-)\vartheta_2(u_2|2i\mathfrak{b}_+).\end{aligned}$$

The coefficient K_0 equals

$$K_0 = ic \exp(D_- \delta - F_-),$$

where

$$\begin{aligned}D_- &= \frac{1}{2} \int_0^{a^2} \frac{tdt}{\sqrt{t(a^2-t)(b^2-t)(c^2-t)}}, \\ F_- &= \frac{1}{2} \int_{c^2}^{\infty} \left(\frac{t}{\sqrt{t(t-a^2)(t-b^2)(t-c^2)}} - \frac{1}{t} \right) dt.\end{aligned}$$

3 Investigation of a dependence of solution (20) on the parameters of the spectral curve

If $\lambda_0 = 0$, then variables x and t are separated to different phases of solution (20). Therefore, for $\lambda_0 = 0$ an amplitude of solution (20) is a periodic function in x and in t with periods $X = A_+/2$ and $T = A_-/4$. If $\lambda_0 \neq 0$, then the variable t is situated in two phases with periods T and $T' = A_+/(8\lambda_0) = X/(4\lambda_0)$. Thus, if $\lambda_0 \neq 0$, then an amplitude of solution (20) is periodic function only in x . Naturally, if T and T' are commensurate, then an amplitude of solution (20) is a periodic function with respect to x as well as to t .

We have to notice that in any case the crests of two-phase solutions are placed in nodes of some lattice of periods. It follows from properties of hyper-elliptic curve Γ (5) that real vectors \mathbf{U} and \mathbf{V} are linearly independent [50, 62]. Therefore, if $g = 2$, then any vector from \mathbb{R}^2 could be represented as a linear combination of vectors \mathbf{U} and \mathbf{V} . In particular this is true for vectors of periods of Riemann two-dimensional theta function, i.e., for $\mathbf{e}_1 = (1, 0)^t$ and $\mathbf{e}_2 = (0, 1)^t$:

$$\Theta(\mathbf{u} + \mathbf{e}_j) \equiv \Theta(\mathbf{u}).$$

Therefore, for any hyper-elliptic curve of genus $g = 2$ there exist real numbers X_j, T_j such that following equality hold

$$X_j \mathbf{U} + T_j \mathbf{V} = \mathbf{e}_j, \quad j = 1, 2.$$

It follows from this equality and from formula (11) that an amplitude of solution (20) is a periodic function on the plane XOT

$$|p|(x + X_j, t + T_j) \equiv |p|(x, t).$$

This proposition is independent from the fact that two-dimensional theta function is expressed by elliptic functions or not. Since a real vector of initial phase \mathbf{Z} could be decomposed on vectors \mathbf{U} and \mathbf{V} , then a change of initial real phase of two-phase solution leads to the trivial shift of a solution on some vector over the plane XOT , and it has no influence on the behavior of a solution. According to this fact in all solutions (except limits) we consider $\mathbf{Z} = \mathbf{0}$.

It follows from properties of elliptic theta functions and from formula (21) that the function $|p|^2(u_1, u_2)$:

1. is a two-periodic function in u_j ,

$$\begin{aligned} |p|^2(u_1 \pm 2, u_2) &= |p|^2(u_1, u_2 \pm 2) = \\ &= |p|^2(u_1 \pm 2i\mathbf{b}_-, u_2) = |p|^2(u_1, u_2 \pm 2i\mathbf{b}_+) = |p|^2(u_1, u_2); \end{aligned}$$

2. satisfies equalities

$$\begin{aligned} |p|^2(u_1 \pm i\mathbf{b}_-, u_2) &= |p|^2(u_1 \pm 1, u_2) \in \mathbb{R}, \\ |p|^2(u_1, u_2 \pm i\mathbf{b}_+) &= |p|^2(u_1, u_2 \pm 1) \in \mathbb{R}. \end{aligned}$$

Therefore, there exists a real smooth solution of NLS equation with complex initial phase \mathbf{Z} . Let us remark that the necessary condition for reality of solution (20) is following:

$$2\Im \mathbf{Z} = \Im BN, \quad \text{where } N \in \mathbb{Z}^g, \quad \Re BN \in \mathbb{Z}^g.$$

Let us now consider dependencies of the periods X and T from parameters of spectral curve (12). We fix the parameter b because a scale transformation of a spectral curve corresponds to scale transformations of solutions' amplitude

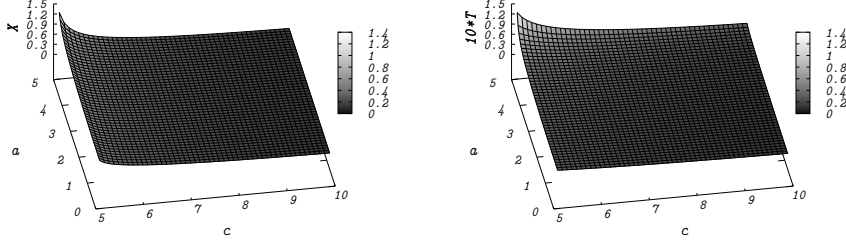


Figure 4: The dependencies of the periods of the phases from a and c when $b = 5$

and periods. It follows from fig. 4 that if a distance between branching points decreases, i.e., if $a \rightarrow b$ or $c \rightarrow b$, then periods X and T grow. In contrary, if a decreases and c grows, then the periods of the phases of solution (20) approach to zero ($X \rightarrow 0$ and $T \rightarrow 0$).

It follows from harmonic analyses that a steepness of solution's fronts depends from existence of a harmonics with high numbers in this solution. The larger is a contribution of harmonics with high numbers the larger a steepness of a solution's front. Evidently, main contribution of a highest harmonics in solution is brought by theta functions, though the certain contribution in solution is given by nonlinearity of the superposition of theta functions. It follows from definition of elliptic Jacobi theta functions that the close to zero are $h_{\pm} = \exp\{-2\pi\mathfrak{b}_{\pm}\}$ the less a contribution of high harmonics of a corresponding phase in solution. The dependence of the quantities h_{\pm} from the parameters of spectral curve (12) is shown on fig. 5. It follows from fig. 5 that for $a/b \rightarrow 1$ the first phase ($\kappa_1 t$) becomes more expressed, and for $a/b \rightarrow 0$ it is less noticeable. Also, if $c \rightarrow b$ and $a \rightarrow b$, then an amplitude of solution is similar to periodically disposed freak waves because a presence of high harmonics in first and second phases.

In the end of this section let us present several figures. The fig. 6 gives an example of a periodic in x and in t solution with behavior of periodically disposed "freak waves" because of both phases of a solution have steep fronts ($a \approx b$ and $c \approx b$). The examples of amplitudes of periodic solutions for $T' = T$ and for different values of parameters of a spectral curve are presented on figures 7 and 8. It is easy to see how a behavior of two-phase solution (20) changes from the case $a \approx b$ and $c \approx b$ (steep fronts, fig. 7) to the case $a \ll b$ and $c \gg b$ (sloping fronts, fig. 8).

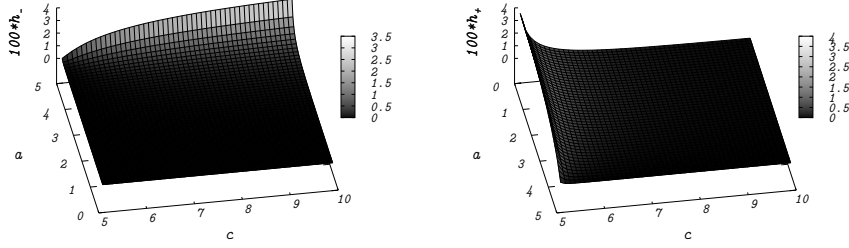


Figure 5: The dependencies of the quantities h_{\pm} from a and c when $b = 5$

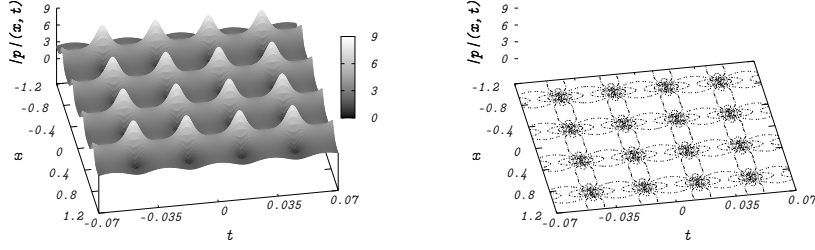


Figure 6: Amplitude of solution (20) for $\lambda_0 = 0$, $a = 6$, $b = 8$, $c = 9$.

4 Simplest limits of solutions

In this section we consider several limits of solution (20); these limits correspond to different confluences of branch points of spectral curve (12).

Let $c \rightarrow b$ in solutions (20). Then a limit of solution (20) has the form of the plane wave:

$$\begin{aligned} p(x, t) \Big|_{c=b} &= 2 \frac{(1 + \sqrt{1 - k_a^2})^2}{2k_a} \cdot \left(\frac{1 + \sqrt{1 - k_a^2}}{k_a} \right)^{-2} \exp(-i\kappa'_1 t) \times \\ &\quad \times \exp\{-2i\lambda_0 x + 2i(-2\lambda_0^2 + a^2 + i\kappa'_1/2)t\} = \\ &= a \exp\{-2i\lambda_0 x - 2i(2\lambda_0^2 - a^2)t\}. \end{aligned} \quad (22)$$

If $a \rightarrow b$, then a limit of solution takes the form of the another plane wave

$$p(x, t) \Big|_{a=b} = c \exp\{-2i\lambda_0 x - 2i(2\lambda_0^2 - c^2)t - i\varphi/2\}. \quad (23)$$

On the other hand, a limit of solution (20) as $a \rightarrow 0$ equals one-phase

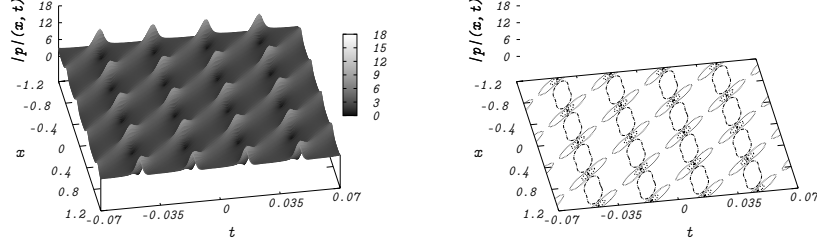


Figure 7: Amplitude of solution (20) for $\lambda_0 = A_+/(2A_-)$, $a = 6$, $b = 8$, $c = 9$.

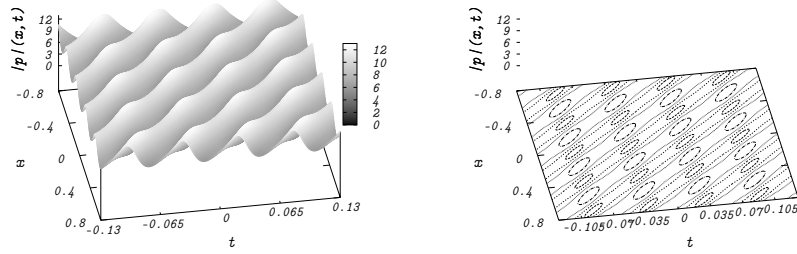


Figure 8: Amplitude of solution (20) for $\lambda_0 = A_+/(2A_-)$, $a = 1$, $b = 3$, $c = 9$.

traveling wave (fig. 9)

$$p(x, t) \Big|_{a=0} = \sqrt{c^2 - b^2} \frac{\vartheta_3(kx + \kappa_2 t | 2ib_+) - \vartheta_2(kx + \kappa_2 t | 2ib_+)}{\vartheta_3(kx + \kappa_2 t | 2ib_+) + \vartheta_2(kx + \kappa_2 t | 2ib_+)} \times \exp\{2i(K_1 x + K_2 t)\}. \quad (24)$$

In order to obtain another representation of solution (24) we rewrite a solution in the form

$$p(x, t) \Big|_{a=0} = f(x + 4\lambda_0 t) e^{-2i\lambda_0 x + 2i(K_{20} - 2\lambda_0^2)t}, \quad f(x) \in \mathbb{R}, \quad K_{20} = b^2 + c^2. \quad (25)$$

Placing this expression into the equation (1) and simplifying it, obtain a differential equation with respect to $f(x)$

$$-2K_{20}f + f_{xx} + 2f^3 = 0$$

or

$$f_x^2 = C + 2K_{20}f^2 - f^4.$$

A solution of last equation is an elliptic Jacobi function [67]. It follows from the fig. 9 that a function $f(x)$ from (25) equals

$$f(x) = \text{Adn}(B(x - x_0); \tilde{k}),$$

where parameters A , B and \tilde{k} satisfy the relations

$$A = B = \sqrt{\frac{2 - \tilde{k}^2}{2K_{20}}}, \quad C = \tilde{k}^2 - 1.$$

Therefore, the solution (24) may be rewritten in the form (see also [21, 22])

$$p(x, t) = \text{Adn}(A(x + 4\lambda_0 t - x_0); \tilde{k}) e^{-2i\lambda_0 x + 2i(K_{20} - 2\lambda_0^2)t}.$$

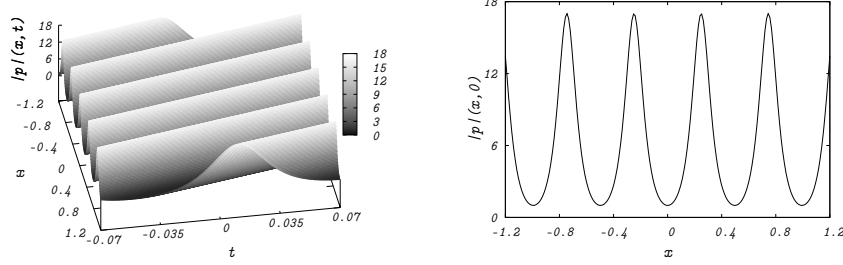


Figure 9: The amplitude of the degenerated solution (24) when $\lambda_0 = 1$, $b = 8$, $c = 9$.

It is not difficult to see that limits of solutions correspond to solutions that is constructed on limits of spectral curves. The solution (22) corresponds to the rational spectral curve

$$w^2 = (\lambda^2 - 2\lambda_0\lambda + |\lambda_1|^2),$$

and (23) corresponds to the curve (also of genus $g = 0$)

$$w^2 = (\lambda^2 - 2\lambda_0\lambda + |\lambda_3|^2).$$

The one-phase solution (24) is associated with the curve of genus $g = 1$

$$w^2 = (\lambda^2 - 2\lambda_0\lambda + |\lambda_2|^2)(\lambda^2 - 2\lambda_0\lambda + |\lambda_3|^2).$$

Some technical details of related calculations can be found in Appendix.

Concluding remarks

From the present work and from previous papers [58, 59] it follows that: a) if an involution $\tau : (w, \lambda) \rightarrow (w, -\lambda)$ exists on a spectral curve $\Gamma = \{(w, \lambda)\}$, then two-gap solution of NLS equation is a periodic in x and in t function; b) if a distance between branching points decreases, then a steepness of front of solution increases. Let us remark that two-phase periodic in x and in t solutions from the present work and from [59] are defined by different formulae; also these solutions have different shapes. The analysis of simplest limits of smooth multi-phase solutions (10) of NLS equation show us that simplest confluences of branching points lead to solutions in the form of plane waves or traveling waves. In order to obtain more interesting solutions it is necessary to consider scaling limits with confluence several pairs of branching points.

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A Asymptotic of the parameters of the solutions

A.1 Limit of (20) as $c \rightarrow b$

Let $a = k_a b$, $0 < k_a < 1$, $c = b + \epsilon$, $\epsilon \rightarrow +0$. Then

$$\begin{aligned}
 A_+ &= \frac{2K\left(\sqrt{\frac{b^2 - a^2}{c^2 - a^2}}\right)}{\sqrt{c^2 - a^2}} \sim \frac{-1}{b\sqrt{1 - k_a^2}} \ln \frac{\epsilon}{8b(1 - k_a^2)} \rightarrow +\infty, \\
 B_+ &= \frac{2K\left(\sqrt{\frac{c^2 - b^2}{c^2 - a^2}}\right)}{\sqrt{c^2 - a^2}} \sim \frac{\pi}{b\sqrt{1 - k_a^2}}, \\
 A_- &= \frac{2K\left(\frac{a}{b}\sqrt{\frac{c^2 - b^2}{c^2 - a^2}}\right)}{b\sqrt{c^2 - a^2}} \sim \frac{\pi}{b^2\sqrt{1 - k_a^2}}, \\
 B_- &= \frac{2K\left(\frac{c}{b}\sqrt{\frac{b^2 - a^2}{c^2 - a^2}}\right)}{b\sqrt{c^2 - a^2}} \sim \frac{-1}{b^2\sqrt{1 - k_a^2}} \ln \frac{k_a^2 \epsilon}{8b(1 - k_a^2)} \rightarrow +\infty,
 \end{aligned}$$

$$\begin{aligned}
B_-^1 &= \frac{2F\left(\frac{b}{c}, \frac{c}{b}\sqrt{\frac{b^2-a^2}{c^2-a^2}}\right)}{b\sqrt{c^2-a^2}} = \\
&= \frac{2\left[K\left(\frac{c}{b}\sqrt{\frac{b^2-a^2}{c^2-a^2}}\right) - F\left(\frac{\sqrt{c^2-a^2}}{c}, \frac{c}{b}\sqrt{\frac{b^2-a^2}{c^2-a^2}}\right)\right]}{b\sqrt{c^2-a^2}} \sim \\
&\sim \frac{-1}{b^2\sqrt{1-k_a^2}} \ln \frac{k_a(1+\sqrt{1-k_a^2})\epsilon}{8b(1-k_a^2)} \rightarrow +\infty, \\
D_- &= \frac{c^2\left[K\left(\frac{a}{b}\sqrt{\frac{c^2-b^2}{c^2-a^2}}\right) - \Pi\left(\frac{a^2}{a^2-c^2}, \frac{a}{b}\sqrt{\frac{c^2-b^2}{c^2-a^2}}\right)\right]}{b\sqrt{c^2-a^2}} \sim \frac{\pi(1-\sqrt{1-k_a^2})}{2\sqrt{1-k_a^2}}, \\
F_- &\sim \ln \frac{2}{1+\sqrt{1-k_a^2}} + \frac{b^2}{2}B_-^1,
\end{aligned}$$

where $K(m)$, $F(m)$ and $\Pi(m)$ are canonical elliptic integrals.

Therefore, the parameters of the solution (20) have following asymptotic as $c \rightarrow b$:

$$\begin{aligned}
\mathbf{b}_- &= \frac{B_-}{A_-} \rightarrow +\infty, \quad \kappa_1 \sim \frac{4b^2\sqrt{1-k_a^2}}{\pi}, \quad \delta \sim \mathbf{b}_- - \frac{2}{\pi} \ln \frac{1+\sqrt{1-k_a^2}}{k_a}, \\
\mathbf{b}_+ &= \frac{B_+}{A_+} \rightarrow 0, \quad k = \frac{2}{A_+} \rightarrow 0, \quad \kappa_2 = \frac{8\lambda_0}{A_+} \rightarrow 0, \\
K_0 &\sim i \frac{(1+\sqrt{1-k_a^2})^2}{2k_a} \exp(-\pi\mathbf{b}_-/2) \rightarrow 0.
\end{aligned}$$

Passing to theta functions that define the first phase of solution (20) we obtain

$$\begin{aligned}
\vartheta_3(\kappa_1 t | 2i\mathbf{b}_-) &\sim 1, \quad \vartheta_2(\kappa_1 t | 2i\mathbf{b}_-) \sim 2h_- \cos(\kappa'_1 t), \\
\vartheta_3(\kappa_1 t + i\delta | 2i\mathbf{b}_-) &\sim 1 + \left(\frac{1+\sqrt{1-k_a^2}}{k_a}\right)^{-4} e^{-2i\kappa'_1 t}, \\
\vartheta_2(\kappa_1 t + i\delta | 2i\mathbf{b}_-) &\sim h_-^{-1} \left(\frac{1+\sqrt{1-k_a^2}}{k_a}\right)^{-2} e^{-i\kappa'_1 t}, \\
\vartheta_3(\kappa_1 t - i\delta | 2i\mathbf{b}_-) &\sim 1 + \left(\frac{1+\sqrt{1-k_a^2}}{k_a}\right)^{-4} e^{2i\kappa'_1 t}, \\
\vartheta_2(\kappa_1 t - i\delta | 2i\mathbf{b}_-) &\sim h_-^{-1} \left(\frac{1+\sqrt{1-k_a^2}}{k_a}\right)^{-2} e^{i\kappa'_1 t},
\end{aligned}$$

where $h_- = \exp(-\pi\mathbf{b}_-/2)$, $\kappa'_1 = 4b^2\sqrt{1-k_a^2}$.

To obtain an asymptotic of second phase we use following relations [67]:

$$\begin{aligned}\vartheta_2(u|i\mathbf{b}) &= \frac{1}{\sqrt{\mathbf{b}}} e^{-\pi u^2/\mathbf{b}} \vartheta_4\left(\frac{i u}{\mathbf{b}} \middle| \frac{i}{\mathbf{b}}\right), \\ \vartheta_3(u|i\mathbf{b}) &= \frac{1}{\sqrt{\mathbf{b}}} e^{-\pi u^2/\mathbf{b}} \vartheta_3\left(\frac{i u}{\mathbf{b}} \middle| \frac{i}{\mathbf{b}}\right)\end{aligned}\tag{26}$$

Let us set a nonzero initial phase $\mathbf{Z} = (0, 1/4)^t$. As the result we get

$$\begin{aligned}\vartheta_3(kx + \kappa_2 t + 2Z_2 \pm 1|2i\mathbf{b}_+) &= \vartheta_3(kx + \kappa_2 t + 2Z_2|2i\mathbf{b}_+), \\ \vartheta_2(kx + \kappa_2 t + 2Z_2 \pm 1|2i\mathbf{b}_+) &= -\vartheta_2(kx + \kappa_2 t + 2Z_2|2i\mathbf{b}_+), \\ \vartheta_3(kx + \kappa_2 t + 2Z_2|2i\mathbf{b}_+) &\sim h_+ \left(1 + e^{k'(x+4\lambda_0 t)}\right), \\ \vartheta_2(kx + \kappa_2 t + 2Z_2|2i\mathbf{b}_+) &\sim h_+ \left(1 - e^{k'(x+4\lambda_0 t)}\right),\end{aligned}$$

where

$$h_+ = \frac{1}{\sqrt{2\mathbf{b}_+}} \exp\left\{-\frac{\pi}{2\mathbf{b}_+}(kx + \kappa_2 t + 2Z_2)^2\right\}, \quad k' = 2b\sqrt{1 - k_a^2}.$$

It can easily be checked that

$$K_1 \sim -\lambda_0, \quad K_2 \sim -2\lambda_0^2 + a^2 + 2b^2\sqrt{1 - k_a^2}.$$

A.2 Limit of (20) as $a \rightarrow b$

If $a \rightarrow b$, then asymptotic of integrals describes by next relations:

$$\begin{aligned}A_+ &\sim \frac{1}{\sqrt{c^2 - b^2}} \int_{a^2}^{b^2} \frac{dt}{\sqrt{(t - a^2)(b^2 - t)}} = \frac{\pi}{\sqrt{c^2 - b^2}}, \\ B_+ &\sim \int_{b^2}^{c^2} \frac{dt}{(t - b^2)\sqrt{(c^2 - t)}} \rightarrow +\infty, \\ A_- &\sim \int_0^{b^2} \frac{dt}{(b^2 - t)\sqrt{t(c^2 - t)}} \rightarrow +\infty, \\ B_- &\sim \frac{1}{b\sqrt{c^2 - b^2}} \int_{a^2}^{b^2} \frac{dt}{\sqrt{(t - a^2)(b^2 - t)}} = \frac{\pi}{b\sqrt{c^2 - b^2}}, \\ B_-^1 &\sim \int_{c^2}^{\infty} \frac{dt}{(t - b^2)\sqrt{t(t - c^2)}} = \frac{1}{b\sqrt{c^2 - b^2}} \arccos \frac{c^2 - 2b^2}{c^2}, \\ D_- &\sim \frac{1}{2} \int_0^{b^2} \frac{tdt}{(b^2 - t)\sqrt{t(c^2 - t)}} = \frac{b^2}{2} A_- - \frac{1}{2} \arccos \frac{c^2 - 2b^2}{c^2}, \\ F_- &\sim \frac{1}{2} \int_{c^2}^{\infty} \left(\frac{1}{\sqrt{t(t - c^2)}} - \frac{1}{t} \right) dt + \frac{b^2}{2} \int_{c^2}^{\infty} \frac{dt}{(t - b^2)\sqrt{t(t - c^2)}} =\end{aligned}$$

$$= \ln 2 + \frac{b^2}{2} B_-^1.$$

Therefore, in this case we have:

$$\begin{aligned} \mathbf{b}_- = \frac{B_-}{A_-} \rightarrow 0, \quad \kappa_1 = \frac{4}{A_-} \rightarrow 0, \quad \delta = \frac{B_-^1}{A_-} \rightarrow 0, \quad \mathbf{b}_+ = \frac{B_+}{A_+} \rightarrow +\infty, \\ k \sim \frac{2\sqrt{c^2 - b^2}}{\pi}, \quad \kappa_2 \sim \frac{8\lambda_0\sqrt{c^2 - b^2}}{\pi}, \quad K_0 \sim i\frac{c}{2}. \end{aligned}$$

Asymptotic of theta functions that define first phase of solution (20) describes by following relations

$$\begin{aligned} \vartheta_3(\kappa_1 t + 2Z_1 | 2i\mathbf{b}_-) &\sim \tilde{h}_-(t) \left(1 + e^{\tilde{\kappa}_1 t}\right), \\ \vartheta_2(\kappa_1 t + 2Z_1 | 2i\mathbf{b}_-) &\sim \tilde{h}_-(t) \left(1 - e^{\tilde{\kappa}_1 t}\right), \\ \vartheta_3(\kappa_1 t + 2Z_1 + i\delta | 2i\mathbf{b}_-) &\sim \tilde{h}_-(t + iB_-^1/4) \left(1 + e^{\tilde{\kappa}_1 t + i\varphi}\right), \\ \vartheta_2(\kappa_1 t + 2Z_1 + i\delta | 2i\mathbf{b}_-) &\sim \tilde{h}_-(t + iB_-^1/4) \left(1 - e^{\tilde{\kappa}_1 t + i\varphi}\right), \\ \vartheta_3(\kappa_1 t + 2Z_1 - i\delta | 2i\mathbf{b}_-) &\sim \tilde{h}_-(t - iB_-^1/4) \left(1 + e^{\tilde{\kappa}_1 t - i\varphi}\right), \\ \vartheta_2(\kappa_1 t + 2Z_1 - i\delta | 2i\mathbf{b}_-) &\sim \tilde{h}_-(t - iB_-^1/4) \left(1 - e^{\tilde{\kappa}_1 t - i\varphi}\right), \end{aligned}$$

where $\tilde{\kappa}_1 = 4b\sqrt{c^2 - b^2}$, initial phase $\mathbf{Z} = (1/4, 0)^t$,

$$\tilde{h}_-(t) = \frac{1}{\sqrt{2\mathbf{b}_-}} \exp \left\{ -\frac{\pi}{2\mathbf{b}_-} (\kappa_1 t + 2Z_1)^2 \right\}, \quad \varphi = \arccos \frac{c^2 - 2b^2}{c^2}.$$

By calculating these asymptotic we use again formulas (26).

Theta-functions that define second phase of solution (20) have the following asymptotic

$$\begin{aligned} \vartheta_3(kx + \kappa_2 t \pm 1 | 2i\mathbf{b}_+) &= \vartheta_3(kx + \kappa_2 t | 2i\mathbf{b}_+), \\ \vartheta_2(kx + \kappa_2 t \pm 1 | 2i\mathbf{b}_+) &= -\vartheta_2(kx + \kappa_2 t | 2i\mathbf{b}_+), \\ \vartheta_3(kx + \kappa_2 t | 2i\mathbf{b}_+) &\sim 1, \\ \vartheta_2(kx + \kappa_2 t | 2i\mathbf{b}_+) &\sim 2\tilde{h}_+ \cos(\tilde{k}x + \tilde{\kappa}_2 t), \end{aligned}$$

where $\tilde{h}_+ = \exp(-\pi\mathbf{b}_+/4)$, $\tilde{k} = 2\sqrt{c^2 - b^2}$, $\tilde{\kappa}_2 = 8\lambda_0\sqrt{c^2 - b^2}$. In addition

$$K_1 \sim -\lambda_0, \quad K_2 \sim -2\lambda_0^2 + c^2.$$

A.3 Limit of (20) as $a \rightarrow 0$

If $a \rightarrow 0$, then elliptic integrals have following asymptotic:

$$A_+ \sim \int_0^{b^2} \frac{dt}{\sqrt{t(b^2 - t)(c^2 - t)}}, \quad B_+ \sim \int_{b^2}^{c^2} \frac{dt}{\sqrt{t(t - b^2)(c^2 - t)}},$$

$$A_- \sim \frac{\pi}{bc}, \quad B_- \sim \int_0^{b^2} \frac{dt}{t\sqrt{(b^2-t)(c^2-t)}} \rightarrow +\infty,$$

$$B_-^1 \sim \frac{1}{bc} \ln \frac{c+b}{c-b}, \quad D_- \sim 0, \quad F_- \sim \frac{1}{2} \ln \frac{4c^2}{c^2-b^2}.$$

From this asymptotic it follows that calculated on curve Γ_- parameters have a form:

$$\mathbf{b}_- = \frac{B_-}{A_-} \rightarrow +\infty, \quad \kappa_1 \sim \frac{4bc}{\pi}, \quad \delta \sim \frac{1}{\pi} \ln \frac{c+b}{c-b}, \quad K_0 \sim i \frac{\sqrt{c^2-b^2}}{2}.$$

Calculated on Γ_- theta functions have following asymptotic

$$\begin{aligned} \vartheta_3(\kappa_1 t | 2i\mathbf{b}_-) &\sim 1, \\ \vartheta_2(\kappa_1 t | 2i\mathbf{b}_-) &\sim 2h_- \cos(4bct), \\ \vartheta_3(\kappa_1 t + i\delta | 2i\mathbf{b}_-) &\sim 1, \\ \vartheta_2(\kappa_1 t + i\delta | 2i\mathbf{b}_-) &\sim h_- \frac{(c^2 + b^2) \cos(4bct) - 2ibc \sin(4bct)}{c^2 - b^2}, \\ \vartheta_3(\kappa_1 t - i\delta | 2i\mathbf{b}_-) &\sim 1, \\ \vartheta_2(\kappa_1 t - i\delta | 2i\mathbf{b}_-) &\sim h_- \frac{(c^2 + b^2) \cos(4bct) + 2ibc \sin(4bct)}{c^2 - b^2}. \end{aligned}$$

Calculated on curve Γ_+ parameters have standard form with substitution $a = 0$; calculated on Γ_- theta functions are standard elliptic theta functions.

In addition

$$K_1 \sim -\lambda_0, \quad K_2 \sim -2\lambda_0^2 + b^2 + c^2.$$

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